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MULTIBODY SYSTEMS WITH UNILATERAL CONSTRAINTS†

F. PFEIFFER

Munich, Germany

e-mail: pfeiffer@lbm.mw.tu.muenchan.de

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A brief survey of theoretical and practical applications of the complementarity principle in multibody systems with many multiple unilateral contacts is presented. Considered in a straightforward manner such systems involve a combinatorial problem of many dimensions, which can only be solved reasonably by the introduction of the complementarity idea. This states that for unilateral contacts either the relative kinematics are zero and the corresponding constraint forces are non-zero, or vice versa. Relations between the complementarity problem and linear programming problems are discussed.

1. CONTACT LAWS AND COMPLEMENTARITY

Complementarity features can be found in many fields of physics, mathematics, especially mathematical optimization, and in areas like operations research and economics. In a these cases two magnitudes or two groups of magnitudes exclude each other. In mechanics all unilateral contacts possess such a property, where either the magnitudes of the relative kinematics are zero and the corresponding constraints forces are non-zero, or vice versa. The product of these two groups of magnitudes is therefore always zero. This constitutes a rule, which enables us to investigate multibody systems with unilateral constraints.

The basic ideas of unilateral mechanics are very old. Fourier describes [1] a "principle of virtual velocity" not only for bilateral but also for unilateral constraints. He defines (in words) the impenetrability condition by considering the properties of relative kinematics in the contact and establishes the principle of virtual work, or better, virtual power in the form used today (see [2]). If we write down his statement in modern mathematical notation, we obtain a set of inequalities defining the complementarity problem.

At the beginning of the twentieth century Boltzmann considered the topic of unilateral contacts in his lectures [3]. He defined the principles of mechanics, virtual work, for example, regarding also unilateral contacts and obtained sets of inequalities. Later Signorini published a paper on problems of elastomechanics, where he introduced an impenetrability condition in the form of a linear complementarity problem, in a form we still use today.

The father of non-smooth mechanics is Moreau, who not only established the mechanical but also the mathematical fundamentals of this new science, which represents a substantial extension of classical mechanics [4, 5]. Panagiotopoulos completed the new theories by introducing inequalities which include non-convex features [6]. Both these scientists apply the idea of complementarity as one important and basic element for their theoretical evolution. Most of their applications refer to problems of elastomechanics.

From the very beginning it was evident, that the methods used by Moreau and Panagiotopoulos could also be transferred to multibody dynamics. After long and fruitful discussions with Panagiotopoulos, who died at an early age in 1998, workers in the Department of Applied Mechanics at the Technical University of Munich about ten years ago started to establish a theory for multibody systems with multiple unilateral contacts, which has since proved its reliability in many theoretical and practical tests [7].

Principally, we may apply two models of contacts. The first model is classical and discretizes the local stiffness and damping behaviour in a suitable way so to be as close as possible to the physical manifestations. The contact formulae of Hertz or finite elements models are examples [8]. The second method ignores local details and approximates contact behaviour by more global and mostly simple relations like Coulomb's law of friction or the impact laws of Newton and Poisson [9]. It is a representation by parametric contact coefficients, which must be measured, and describes the most

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important relations, which hold at least for a large variety of applications. We shall focus on the second method, which also fits better into the framework of multibody dynamics.

First we will recall the well-known laws mentioned above. For dry friction we shall apply Coulomb's law in the following form

$$\begin{aligned} &|\lambda_{T_i}| < \mu_{0i}\lambda_{N_i}, \quad \dot{g}_{T_i} = 0 \quad \text{(sticking)} \\ &\lambda_{T_i} = +\mu_{0i}\lambda_{N_i}, \quad \dot{g}_{T_i} \leq 0 \quad \text{(negative sliding)} \\ &\lambda_{T_i} = -\mu_{0i}\lambda_{N_i}, \quad \dot{g}_{T_i} \geq 0 \quad \text{(positive sliding)} \end{aligned} \tag{1.1}$$

where $\dot{g}_{T_i}$ is the relative velocity at contact i, and λ_{N_i} and λ_{T_i} are the relevant constraint forces in the normal and tangential directions respectively. Equation (1.1) can be interpreted as a double corner law as shown in Fig. 1, where the segments marked 1, 2 and 3 correspond to sticking, negative sliding, and positive sliding respectively. The quantity $\lambda_{0i} = \mu_{0i}\lambda_{N_i} - |\lambda_{T_i}|$ will be called "friction saturation".

We are either within the friction cone with

$$|\dot{g}_{T_i}| = 0, \quad -\mu_{0i}\lambda_{N_i} \leq \lambda_{T_i} \leq \mu_{0i}\lambda_{N_i}$$

or on its surface where

$$|\lambda_{T_i}| = \mu_{0i}\lambda_{N_i}$$

The friction coefficient μ_{0i} is defined as the limit (see Fig. 2)

$$\mu_{0i} = \lim_{\dot{g}_{T_i} \to 0} \mu_i(\dot{g}_{T_i}) \tag{1.2}$$

The constraint force λ_{N_i} in the normal direction results also from a contact law, which might be characterized by a contact separation mechanism. If we have a normal relative distance at contact i,

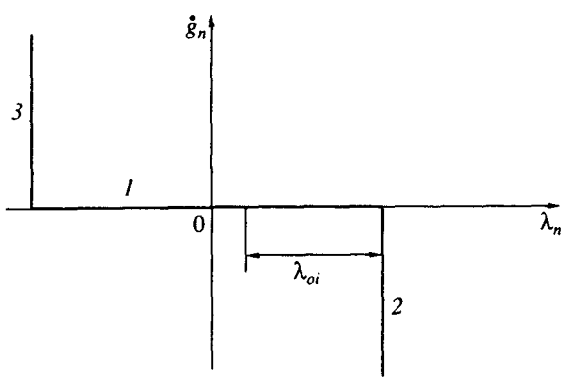

Fig. 1

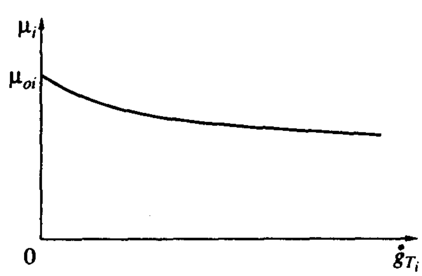

Fig. 2

denoted by g_{N_i}, then the interdependency with the corresponding constraint force λ_{N_i} consists of the classical complementarity principle [1, 10]: either $g_{N_i} = 0$, $\lambda_{N_i} \geqslant 0$ (half-line 1 in Fig. 3), or $g_{N_i} \geqslant 0$, $\lambda_{N_i} = 0$ (half-line 2). In this case the product $g_{N_i}\lambda_{N_i}$ is always equal to zero.

Both contact laws (Figs 1 and 3) include complementarity features, because the double corner in Fig. 1 can be decomposed in two so-called "unilateral primitives" in the form of two simple corners. We can say in the case of frictional contacts that either the relative velocity $\dot{g}_{T_i}$ is zero and the friction saturation λ_{0i} is non-zero, or vice versa; the product $\dot{g}_{T_i}\lambda_{0i}$ is always zero.

As regards the impact laws, we have in classical mechanics two models, Newton's kinematic and Poisson's kinetic relations. Newton's law connects the relative normal velocity after an impact with that before the impact, stating

$$\dot{g}^+_{N_i} = -\varepsilon_i \dot{g}^-_{N_i} \tag{1.3}$$

where minus and plus superscripts correspond to instants shortly before and shortly after the impact. Energy losses are taken into account by the coefficient of restitution ε_i.

Poisson's law determines the relation between the impulses $\Lambda = \int \lambda dt$ in the form

$$\Lambda^+_i = \varepsilon^*_i \Lambda^-_i \tag{1.4}$$

The physical idea behind Eq. (1.4) consists of the storage of impulses during compression and a gain connected with losses during expansion of the impact process. Therefore, Poisson's law can be applied in the normal and tangential direction's of the contact without generating physical inconsistency. In any case the loss coefficients ε_i, ε^*_i have to be measured for each specific pair of materials.

Consider an example of the combinatorial problem, connected with many contacts, and its solution. A continuously variable transmission changes the transmission ratio continuously by reducing hydraulically the axial distance of the conical disc on one side and at the same time increasing the axial distance of the second conical disc [11]. The chain itself consists of elements with rocker pins, which come into contact with the disc. Each rocker pin, being in contact with the two sides of the disc, can move in radial and circumferential directions or stick, which results in three possibilities. Next we may have a contact or a detachment state, which gives another two possibilities. If we have ten rocker pin elements within the conical disc of one wheel only, we obtain 5^{10} contact combinations. Item-by-item examination of these possibilities is obviously impossible. From the other hand, using complementarity ideas, we can solve the problem in a reasonable time [12].

2. UNILATERAL DYNAMICS

The equations of motion for multibody systems with unilateral contacts are published elsewhere [7, 13]; we shall therefore give only a short summary here. Representing, as a first step only, all bilateral constraints and couplings by force laws, we can always represent multibody systems as follows:

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{0}, \quad \mathbf{q} \in \mathbb{R}^f, \ \mathbf{h} \in \mathbb{R}^f, \ \mathbf{M} \in \mathbb{R}^{f,f}, \tag{2.1}$$

where $\mathbf{q}$ are generalized coordinates, $\mathbf{M}$ is a symmetric mass matrix, and $\mathbf{h}$ are all forces acting in the system. Unilateral contacts do not block any further degree of freedom. If some of the unilateral constraints become active, the remaining number of degrees of freedom is smaller than f. Furthermore, contact geometry for spatial cases has been considered by means of differential geometry and corresponding parametrization [14, 15].

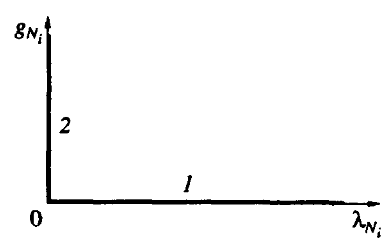

Fig. 3

We first define all time-dependent contact sets present in a unilateral multibody system [7]. The set I_A consists of n_A indices of all contact points. The elements of the set I_C are the n_C indices of the unilateral constraints with vanishing normal distance $g_{N_i} = 0$, but arbitrary relative velocity in the normal direction. The index set $I_N \subseteq I_C$ contains the n_N indices of the potentially active normal constraints which satisfy the necessary conditions for continuous contact ($g_{N_i} = 0, \dot{g}_{N_i} = 0$). This set includes, for example, all contact states with slipping. The n_T elements of the set $I_T \subset I_N$ are the indices of the potentially active tangential constraints with $\dot{g}_{T_i} = 0$. The numbers of elements of these index sets are not constant because there are variable states of constraints due to separation and stick-slip phenomena.

As a next step we must organize all transitions from contact to detachment and from stick to slip and the corresponding reverse transitions. In the normal direction of a contact we find the following situations at contact i: passive contact $g_{N_i}(\mathbf{q}, t) \geqslant 0$, $\lambda_{N_i} = 0$ (indicator g_{N_i}), transition to contact, $g_{N_i}(\mathbf{q}, t) = 0$, $\lambda_{N_i} \geqslant 0$, active contact $g_{N_i}(\mathbf{q}, t) = 0$, $\lambda_{N_i} > 0$ (indicator λ_{N_i}, constraint $\ddot{g}_{N_i} = 0$), transition to detachment $g_{N_i}(\mathbf{q}, t) \geqslant 0$, $\lambda_{N_i} = 0$.

The kinematic quantities $g_{N_i}, \dot{g}_{N_i}, \ddot{g}_{N_i}$ are given from contact geometry. The constraint forces λ_{N_i} must be compressive forces. If they change sign, we get separation. Thus, we have n_N complementarity conditions (expressed on an acceleration level)

$$\ddot{g}_N \geqslant 0, \ \ \lambda_N \geqslant 0, \ \ \ddot{g}_N^T \lambda_N = 0 \tag{2.2}$$

They are equivalent to the variational inequality

$$-\ddot{g}_N^T(\lambda_N^* - \lambda_N) \leqslant 0, \ \ \lambda_N \in C_N, \ \ \forall \lambda_N^* \in C_N \tag{2.3}$$

The convex set $C_N = \{\lambda_N^* : \lambda_N^* \geqslant 0\}$ contains all admissible contact forces. Conditions (2.2) correspond to Fig. 3.

With respect to the tangential direction of a contact we shall confine our considerations to the application of Coulomb's friction law, which in no way means a loss of generality. The complementary behaviour is a characteristic feature of all contact phenomena, irrespective of the specific physical law of contact. Furthermore, we shall assume that within the infinitesimal time step for a transition from stick to slip and vice versa the coefficients of static and sliding friction are the same, which is expressed by Eq. (1.2). For $\dot{g}_{T_i} \neq 0$ any friction law may be applied (see Fig. 2). With this property Coulomb's friction law distinguishes between the two cases

$$\begin{aligned} &|\lambda_{T_i}| < \mu_{0i}\lambda_{N_i} \Rightarrow |\dot{g}_{T_i}| = 0, \ \ i \in I_T \\ &|\lambda_{T_i}| = \mu_{0i}\lambda_{N_i} \Rightarrow |\dot{g}_{T_i}| > 0, \ \ i \in I_N \setminus I_T \end{aligned} \tag{2.4}$$

The first formula in (2.4) corresponds to stiction; this means that if the relative tangential velocity is zero, the constraint force lies within the friction cone. The second formula corresponds to sliding; this means that the constraint force lies on the friction cone. In addition we must take into account the fact that in the tangential contact plane we might get one or two directions depending on plane or spatial contact.

From this we can summarize the possibilities in the tangential direction: passive contact (sliding, set $i \in I_N \setminus I_T$, indicator $|\dot{g}_{T_i}| = 0$); transition from slip to stick; active contact (sticking, $i \in I_T$, indicator $|\mu_{0i}\lambda_{N_i}| - |\lambda_{T_i}| \geqslant 0$, constraint $\ddot{g}_{T_i} = 0$); transition from stick to slip.

From a numerical point of view, we have to check the indicator for a change in sign, which then requires subsequent interpolation. For a transition from stick to slip one must examine the possible development of a non-zero relative tangential acceleration at the start of sliding.

Equation (2.4) put on an acceleration level can then be written in the more detailed form

$$\begin{aligned} &|\lambda_{T_i}| < \mu_{0i}\lambda_{N_i}, \ \ \ddot{g}_{T_i} = 0 \ \ i \in I_T \ \ \text{(sticking)} \\ &\lambda_{T_i} = +\mu_{0i}\lambda_{N_i}, \ \ \ddot{g}_{T_i} \leqslant 0 \ \ i \in I_N \setminus I_T \ \ \text{(negative sliding)} \\ &\lambda_{T_i} = -\mu_{0i}\lambda_{N_i}, \ \ \ddot{g}_{T_i} \geqslant 0 \ \ i \in I_N \setminus I_T \ \ \text{(positive sliding)} \end{aligned} \tag{2.5}$$

Equation (2.5) corresponds to Fig. 1, which can be decomposed into two or four simple corners (see Fig. 3) by proper choice of variables. We obtain [16]

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}, \ \mathbf{y} \geq 0, \ \mathbf{x} \geq 0, \ \mathbf{y}^T\mathbf{x} = 0, \ \mathbf{y}, \mathbf{x} \in \mathbb{R}^{n^*} \tag{2.6}$$

where $n^* = n_N + 4n_T$ in the case of decomposition into four and $n^* = n_N + 2n_T$ for decomposition into two elementary corners. The quantity $\mathbf{x}$ includes the contact forces and one part of the decomposed accelerations, and the quantity $\mathbf{y}$ includes the relative accelerations and in addition the friction saturations λ_{0i} (see Fig. 1). Equation (2.6) describes a linear complementarity problem, thus being adequate for plane contacts. In the case of spatial contacts the friction saturation contains the geometric sum of two possible friction directions, leading to a non-linearity which cannot be solved in a straightforward way [17].

Just as in the normal case we can represent contact law (2.5) by a variational inequality of the form

$$\ddot{g}_{T_i}^T(\lambda_{T_i}^* - \lambda_{T_i}) \geq 0, \quad \lambda_{T_i} \in C_{T_i}, \quad \forall \lambda_{T_i}^* \in C_{T_i} \tag{2.7}$$

$$C_{T_i} = \left\{ \lambda_{T_i}^* : \left| \lambda_{T_i}^* \right| \leq \mu_{0i}\lambda_{N_i}, \ i \in I_T \right\}$$

The convex set C_{T_i} contains all admissible contact forces $\lambda_{T_i}^*$ in the tangential direction.

To derive the equations of motion including unilateral effects we must combine the multibody equations (2.1) with the unilateral constraints (2.2) and (2.6). As a first step we include the constraint forces in Eq. (2.5), keeping in mind that in a system with additional unilateral constraints the number of degrees of freedom is variable. To avoid difficulties with many different sets of minimal coordinates, we take one set of generalized coordinates for each of combinations $I_A \setminus I_C$, $I_A \setminus I_N$, $I_A \setminus I_T$, and include the active unilateral constraints and their constraint forces into the equations of motion by a Lagrange multiplier technique.

$$\begin{gathered} \mathbf{M}\ddot{\mathbf{q}} - \mathbf{h} - (\mathbf{W} + \mathbf{N}_G)\boldsymbol{\lambda} = 0, \quad \ddot{\mathbf{g}} = \mathbf{W}^T\ddot{\mathbf{q}} + \overline{\mathbf{w}} \\ \ddot{\mathbf{g}} = \begin{Vmatrix} \ddot{\mathbf{g}}_N \\ \ddot{\mathbf{g}}_T \end{Vmatrix}, \quad \boldsymbol{\lambda} = \begin{Vmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \end{Vmatrix}, \quad \overline{\mathbf{w}} = \begin{Vmatrix} \overline{\mathbf{w}}_N \\ \overline{\mathbf{w}}_T \end{Vmatrix} \\ \mathbf{W} = (\mathbf{W}_N, \mathbf{W}_T), \quad \mathbf{N}_G = (\mathbf{H}_R, 0) \end{gathered} \tag{2.8}$$

The subscripts N and T stand for the normal and tangential direction (the plane case), $\mathbf{W}$ are the unilateral constraint matrices and $\mathbf{N}_G$ represents contacts with sliding friction. Outside transition events and thus for an unchanging contact configuration the relative accelerations $\ddot{\mathbf{g}}$ are zero. In this case Eqs (2.8) have a solution for $\ddot{\mathbf{g}}$, λ.

We now combine the equations of motion in the form (2.8) with variational inequalities (2.3) and (2.7). The resulting system is not solvable. The variational inequalities are therefore converted into equalities. From this we get a non-linear system of equations which represents linear or non-linear complementarity problems depending on the type of contact, i.e. plane or spatial contacts. Linear complementarity problems can be solved by algorithms related to linear programming methods, for example, Lemke's algorithm; non-linear complementarity problems require iterative algorithms [17].

On the basis of the above ideas and before the background of [5], a new theory of impacts with friction has been developed [14] and verified with more than 600 tests on a specially designed impact machine [18]. It has been applied successfully to many practical problems, one of which is the damping of wind-induced oscillations of a stack damper. A numerical analysis, carried out in [19], showed good agreement between the theory and experimental data.

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